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Interaction of a hollow cylinder of finite length and a plate with a cylindrical cavity with a rigid insert☆

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ABSTRACT

Two problems of the interaction of a hollow circular cylinder with load-free ends and an unbounded plate with a cylindrical cavity and a symmetrically imbedded rigid insert are considered. Homogeneous solutions are found and the generalized orthogonality of these solutions is used when the modified boundary conditions are satisfied. As a result, we have a system of two integral equations in functions of the displacements of the outer and inner surfaces of the hollow cylinder. These functions are sought in the form of sums of a trigonometric series and a power function with a root singularity. The ill-posed infinite systems of linear algebraic equations obtained are regularized by the introduction of small positive parameters. Since the elements of the matrices of the systems as well as the contact stresses are defined by poorly converging numerical and functional series, an efficient method for calculating of the remainders of the above-mentioned series is developed. Formulae are found for the contact pressure distribution function and the integral characteristic. Examples of the calculation of the interaction of the cylinder and the plate with an insert are given.

The method of solving contact problems described here has been used earlier[1,2] and the generalized orthogonality of the solutions found for bodies of finite dimensions, that is, for a rectangle and cylinders of finite length, is its basis. Problems for hollow cylinders with a band [2] and an insert reduce to a system of two integral equations, and the problem for a rectangle[1] reduces to one integral equation. Solving these integral equations, ill-posed systems of linear algebraic equations are obtained which are subject to regularization[3].

1. Formulation of the problem and homogeneous solutions

The axisymmetric problem of the interaction of a hollow elastic cylinder of radii R_0, $R_1 (0 < R_0 < R_1)$ and length ($|z| \le 1$) with a symmetrically imbedded rigid insert of a length $2a$ and lateral surface $r = R_0 + \delta(z)$, where $\delta(z)$ is an even function of z (Fig. 1), is considered. We shall assume that there are no friction forces

in the region of contact between the insert and the cylinder and that the ends of the cylinder and its outer surface $r = R_1$ are unloaded. The boundary conditions can then be written in the form

$$\sigma_z(r, \pm 1) = \tau_{rz}(r, \pm 1) = 0, \quad R_0 \le r \le R_1 \tag{1.1}$$

$$\tau_{rz}(R_1, z) = \tau_{rz}(R_0, z) = \sigma_r(R_1, z) = 0, \quad |z| \le 1; \quad u_r(R_0, z) = \delta(z), \quad |z| \le a \tag{1.2}$$

$$\sigma_r(R_0, z) = 0, \quad a < |z| \le 1 \tag{1.3}$$

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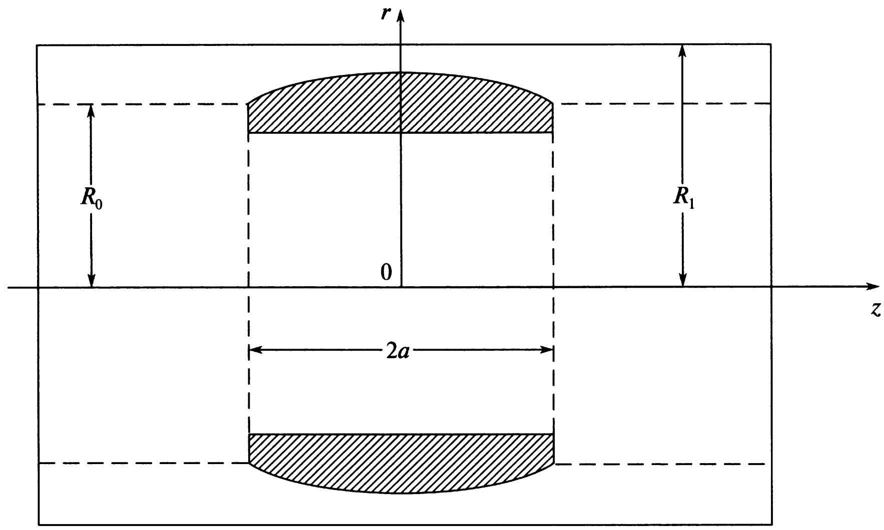

Fig. 1.

We will use of the general representation of the solution of an axisymmetric problem in terms of a biharmonic Love functions $\Phi(r, z)$[4]

$$\Delta^2\Phi = 0,\ \Delta \equiv \partial_r^2 + r^{-1}\partial_r + \partial_z^2,\quad \partial_r \equiv \partial/\partial r,\quad \partial_z \equiv \partial/\partial z,\quad 2Gu_r = -\partial_r\partial_z\Phi$$

$$2Gu_z = [(2-2\nu)\Delta - \partial_z^2]\Phi,\quad \sigma_r = (\nu\Delta - \partial_r^2)\partial_z\Phi,\quad \sigma_z = [(2-\nu)\Delta - \partial_z^2]\partial_z\Phi$$

$$\tau_{rz} = \partial_r[(1-\nu)\Delta - \partial_z^2]\Phi,\quad \sigma_\varphi = (\nu\Delta - r^{-1}\partial_r)\partial_z\Phi \qquad (1.4)$$

where G is the shear modulus and ν is Poisson's ratio.

For a hollow cylinder, we will seek the Love function in the form $\Phi = f^{(0)}(r)\psi(z)$. Here,

$$f^{(s)}(r) = c_1 H_s^{(1)}(\gamma r) + c_2 H_s^{(2)}(\gamma r),\quad s = 0,1$$

$H_s^{(1)}(\gamma r)$ and $H_s^{(2)}(\gamma r)$ are Hankel functions [5] and c_1, c_2, γ are constants. From relations (1.4), we find

$$\Delta^2\Phi \equiv f^{(0)}(r)(\partial_z^2 - \gamma^2)^2\psi(z) = 0,\quad \psi(z) = C_1\operatorname{sh}\gamma z + C_2 z\operatorname{ch}\gamma z \quad (C_1, C_2 - \text{const})$$

$$2Gu_r = \gamma f^{(1)}(r)\psi'(z),\quad 2Gu_z = f^{(0)}(r)(\psi''(z) - 2\chi(z)),\quad \tau_{rz} = \gamma f^{(1)}(r)\chi(z)$$

$$\sigma_z = f^{(0)}(r)\chi^*(z),\quad \sigma_r = f^{(0)}(r)\chi'(z) - r^{-1}\gamma f^{(1)}(r)\psi'(z)$$

$$\chi(z) = \nu\psi''(z) + (1-\nu)\gamma^2\psi(z),\quad \chi^*(z) = (1-\nu)\psi'''(z) - (2-\nu)\gamma^2\psi'(z) \qquad (1.5)$$

Whence, satisfying boundary conditions (1.1), we obtain

$$C_1 = -(1/\operatorname{sh}\gamma_n + 2\nu\beta_n)/2,\quad C_2 = \gamma_n\beta_n/2,\quad \beta_n = (\gamma_n\operatorname{ch}\gamma_n)^{-1}$$

$$\operatorname{sh}2\gamma_n + 2\gamma_n = 0;\quad \operatorname{Re}\gamma_n \geq 0,\quad n = 0,1\ldots \qquad (1.6)$$

Taking account of relations (1.5) and (1.6), we find the eigenfunction $\Psi_n(z)$ and the stress-strain state corresponding to the non-zero eigenvalue $\gamma_n (n = 1, 2, \ldots)$:

$$\Phi_n = f_n^{(0)}(r)\Psi_n(z),\quad \Psi_n(z) = F_n'(z) - \nu\beta_n\operatorname{sh}\gamma_n z,\quad \chi_n(z) = \gamma_n^2 F_n'(z),\quad \chi_n^*(z) = -\gamma_n^4 F_n(z)$$

$$F_n(z) = \frac{1}{2}(z\operatorname{sh}\gamma_n z - \operatorname{ch}\gamma_n z\operatorname{th}\gamma_n)\beta_n = \frac{1}{\gamma_n^2}\left(F_n''(z) - \frac{\operatorname{ch}\gamma_n z}{\operatorname{ch}\gamma_n}\right),\quad \sigma_z^{(n)} = f_n^{(0)}(r)\chi_n^*(z)$$

$$\tau_{rz}^{(n)} = \gamma_n f_n^{(1)}(r)\chi_n(z),\quad \sigma_r^{(n)} = f_n^{(0)}(r)\chi_n'(z) - r^{-1}\gamma_n f_n^{(1)}(r)\Psi_n'(z)$$

$$2Gu_z^{(n)} = f_n^{(0)}(r)(\Psi_n''(z) - 2\chi_n(z)),\quad 2Gu_r^{(n)} = \gamma_n f_n^{(1)}(r)\Psi_n'(z)$$

$$f_n^{(s)}(r) = c_{1,n}H_s^{(1)}(\gamma_n r) + c_{2,n}H_s^{(2)}(\gamma_n r)\ (c_{1,n}, c_{2,n} - \text{const}) \qquad (1.7)$$

The following equations correspond to the eigenvalue $\gamma_0 = 0$:

$$\Phi_0 = c_{1,0}[(2-\nu)z^3/3 - (1-\nu)zr^2/2] + c_{2,0}z\ln r, \quad \sigma_r^{(0)} = c_{1,0}(1+\nu) + c_{2,0}r^{-2}$$

$$\sigma_z^{(0)} = \tau_{rz}^{(0)} = 0,\ 2Gu_r^{(0)} = c_{1,0}(1-\nu)r - c_{2,0}r^{-1}, \quad 2Gu_z^{(0)} = -2\nu c_{1,0}z \tag{1.8}$$

From relations (1.7) and (1.8) when $r = R_s (s = 0, 1)$, we find

$$\tau_{rz}^{(n)}(R_s, z) = f_{n,s}\chi_n(z), \quad 2Gu_r^{(n)}(R_s, z) = f_{n,s}\Psi'_n(z), \quad 2Gu_r^{(0)}(R_s, z) = (1-\nu)f_{0,s}$$

$$f_{0,s} = c_{1,0}R_s - c_{2,0}R_s^{-1}/(1-\nu), \quad \sigma_r^{(n)}(R_s, z) = f_n^{(0)}(R_s)\chi'_n(z) - R_s^{-1}f_{n,s}\Psi'_n(z)$$

$$f_{n,s} = \gamma_n f_n^{(1)}(R_s), \quad 2Gu_z^{(n)}(R_s, z) = f_n^{(0)}(R_s)(\Psi''_n(z) - 2\chi_n(z))$$

$$\sigma_r^{(0)}(R_s, z) = \sum_{h=0}^{1} f_{0,h}A_{0,s}^h, \quad \sigma_z^{(n)}(R_s, z) = f_n^{(0)}(R_s)\chi_n^*(z) \tag{1.9}$$

$$A_{0,s}^s = R_s^{-1}[1 + \nu + 2(-1)^k R_k^2/\Delta_0], \quad A_{0,s}^k = 2(-1)^s R_k/\Delta_0, \quad \Delta_0 = R_1^2 - R_0^2 \quad (k \neq s = 0,1)$$

$$\sigma_r(R_s, z) = \sum_{n=0}^{\infty}{}' \sigma_r^{(n)}(R_s, z), \quad u_r(R_s, z) = \sum_{n=0}^{\infty}{}' u_r^{(n)}(R_s, z), \quad \tau_{rz}(R_s, z) = \sum_{n=1}^{\infty}{}' \tau_{rz}^{(n)}(R_s, z) \tag{1.10}$$

Henceforth a prime on a summation sign denotes a truncated form:

$$\sum_{n=0}^{\infty}{}' G_n(z) \equiv G_0(z) + 2\mathrm{Re}\left\{\sum_{n=1}^{\infty} G_n(z)\right\} \quad (\mathrm{Re}\gamma_n, \mathrm{Im}\gamma_n > 0)$$

In the case of an unbounded plate with a cylindrical cavity when $R_0 = R$, $R_1 = \infty \to (R \le r < \infty)$, the Love function is sought in the form $\Phi = K_0(tr)\psi(z)(t = \mathrm{const})$ and the relations of the form of (1.5) - (1.9) have the form

$$\psi(z) = C_1\sin tz + C_2 z\cos tz, \quad \sin 2t_n + 2t_n = 0; \quad \mathrm{Re}t_n \ge 0 \quad (2t_n = \mu_n - i\zeta_n, \text{ см. } [2])$$

$$\Psi_{ni}(z) = F'_{ni}(z) - \nu\beta_{ni}\sin t_n z, \beta_{ni} = (t_n\cos t_n)^{-1}, \quad \chi_{ni}(z) = -t_n^2 F'_{ni}(z), \quad \chi_{ni}^*(z) = -t_n^4 F_{ni}(z)$$

$$F_{ni}(z) = \frac{1}{2}(z\sin t_n z - \cos t_n z\tan t_n)\beta_{ni} = \frac{1}{t_n^2}\left(\frac{\cos t_n z}{\cos t_n} - F''_{ni}(z)\right), \quad \sigma_r^{(0)}(R, z) = f_0(\nu - 1)R^{-1}$$

$$\sigma_r^{(n)}(R, z) = f_n(t_n^{-1}A_n\chi'_{ni}(z) - R^{-1}\Psi'_{ni}(z)), \quad \sigma_z^{(n)}(R, z) = f_n t_n^{-1}A_n\chi_{ni}^*(z)$$

$$\tau_{rz}^{(n)}(R, z) = f_n\chi_{ni}(z), \quad 2Gu_r^{(0)}(R, z) = (1-\nu)f_0, \quad 2Gu_r^{(n)}(R, z) = f_n\Psi'_{ni}(z)$$

$$2Gu_z^{(n)}(R, z) = f_n t_n^{-1}A_n(\Psi''_{ni}(z) - 2\chi_{ni}(z)), \quad A_n = K_0(t_n R)/K_1(t_n R), \quad n = 1, 2, \ldots \tag{1.11}$$

Here, $K_0(t_n R)$, $K_1(t_n R)$ are McDonald functions and f_0 and f_n are constants.

The eigenfunctions for $\Psi_n(z)$ and $\Psi_{ni}(z)$ for the cylinder and the plate, defined by formulae (1.7) and 91.11), satisfy the conditions of generalized orthogonality [2]

$$\int_{-1}^{1}[F'_n(z)\beta_m \mathrm{sh}\gamma_m z + F'_m(z)\beta_n \mathrm{sh}\gamma_n z]dz = \begin{cases} 0, & m \neq n \\ -\gamma_n^{-2}, & m = n \end{cases} \tag{1.12}$$

$$\int_{-1}^{1}[F'_{ni}(z)\beta_{mi}\sin t_m z + F'_{mi}(z)\beta_{ni}\sin t_n z]dz = \begin{cases} 0, & m \neq n \\ t_n^{-2}, & m = n \end{cases} \tag{1.13}$$

2. Method of solution

We introduce the notation

$$u_r(R_1, z) = u_1(z) \equiv u(z), \ |z| \le 1, \quad u_r(R_0, z) = u_0(z) \equiv \begin{cases} \delta(z), & |z| \le a \\ g(z), & a \le |z| \le 1 \end{cases} \tag{2.1}$$

Here $u(z)$ and $g(z)$ are the sought function which are even in z. The second boundary condition of (1.2), supplemented by the first relation of (2.1), can then be written in the form

$$u_r(R_s, z) = u_s(z), \quad |z| \le 1, \quad s = 0,1 \tag{2.2}$$

Since the functional series (1.10), which determine the left-hand sides of the first condition of (1.2) and conditions (1.3) and (2.2) diverge (an a posteriori analysis of the solution is evidence of this), the above mentioned boundary conditions are replaced by the following boundary conditions

$$\int_0^z\int_1^\eta \tau_{rz}(R_s,\xi)d\xi d\eta \equiv \sum_{n=1}^\infty f_{n,s}(F_n'(z) - \beta_n \mathrm{sh}\gamma_n z) = 0, \quad |z| \le 1, \quad s = 0,1 \tag{2.3}$$

$$2G\int_0^z u_r(R_s,\xi)d\xi \equiv f_{0,s}(1-\nu)z + \sum_{n=1}^\infty f_{n,s}(F_n'(z) - \nu\beta_n \mathrm{sh}\gamma_n z) = 2G\int_0^z u_s(\xi)d\xi \tag{2.4}$$

$$\sigma(R_s,z) \equiv \int_1^z\int_1^\eta\int_1^t \sigma_r(R_s,\xi)d\xi d\eta dt \equiv \sum_{h=0}^1\left\{\frac{1}{2}f_{0,h}A_{0,s}^h f(z) + \frac{1}{4}\sum_{n=1}^\infty f_{n,h}A_{n,s}^h \tilde{F}_n(z)\right\} +$$

$$+\frac{1}{4}c_s\sum_{n=1}^\infty f_{n,s}\tilde{H}_n(z) = 0 \quad \text{when} \quad s = 0,\ a \le z \le 1 \quad \text{when} \quad s = 1,\ 0 \le z \le 1 \tag{2.5}$$

Here,

$$\tilde{F}_n(z) = 4\gamma_n^{-1}\left[F_n'(z) + \beta_n(\mathrm{sh}\gamma_n - \mathrm{sh}\gamma_n z)\right], \quad \tilde{H}_n(z) = \gamma_n^{-2}\left[1 - z + \beta_n(\mathrm{sh}\gamma_n z - \mathrm{sh}\gamma_n)\right]$$

$$A_{n,s}^s = \frac{H_0^{(s+1)}(R_s\gamma_n) - H_1^{(s+1)}(R_k\gamma_n)A_{n,s}^k}{H_1^{(s+1)}(R_s\gamma_n)} - \frac{1}{R_s\gamma_n}, \quad A_{n,s}^k = (-1)^s\frac{4}{\pi i R_s\gamma_n\Delta_n}$$

$$\gamma_n f_n^{(0)}(R_s) - \frac{1}{R_s\gamma_n}f_{n,s} = \sum_{h=0}^1 f_{n,h}A_{n,s}^h, \quad c_s = \frac{4\nu}{R_s}, \quad f(z) = \frac{(z-1)^3}{3}$$

$$\Delta_n = H_1^{(1)}(R_0\gamma_n)H_1^{(2)}(R_1\gamma_n) - H_1^{(1)}(R_1\gamma_n)H_1^{(2)}(R_0\gamma_n), \quad n = 1,2,\ldots; \quad k \ne s = 0,1$$

Equations (2.3) and (2.4) are equivalent to the system of relations

$$\sum_{n=1}^\infty f_{n,s}F_n'(z) = 2\theta\int_0^z u_s(\xi)d\xi - f_{0,s}z, \quad \sum_{n=1}^\infty f_{n,s}\beta_n \mathrm{sh}\gamma_n z = 2\theta\int_0^z u_s(\xi)d\xi - f_{0,s}z, \quad |z| \le 1 \tag{2.6}$$

Here,

$$f_{0,s} = 2\theta\int_0^1 u_s(\xi)d\xi, \quad \theta = \frac{G}{1-\nu} \tag{2.7}$$

We next determine the constants $f_{n,s}$ using the generalized orthogonality condition (1.12). Multiplying the first equation of (2.6) by $\beta_m sh\gamma_m z$ and the second by $F_m'(z)$ and then adding and integrating with respect to z, we find

$$f_{n,s} = 4\theta\int_0^1 u_s(\xi)F_n''(\xi)d\xi, \quad s = 0,1; \quad n = 1,2,\ldots \tag{2.8}$$

Replacing the coefficients $f_{0,s}, f_{1,s}, f_{2,s}, \ldots$ in relation (2.5) by integrals (2.7) and (2.8) and taking account of equality (2.1), we give condition (2.5) the form

$$\sigma(R_s,z) = \theta\left\{\int_0^1 u(\xi)K_{1,s}d\xi + \int_a^1 g(\xi)K_{0,s}d\xi + \int_0^a \delta(\xi)K_{0,s}d\xi\right\} = 0 \tag{2.9}$$

when $s = 0$, $a \le z \le 1$ and, when $s = 1$, $0 \le z \le 1$where

$$K_{h,s} = f_{h,s}(z) + \sum_{n=1}^\infty F_n''(\xi)\Psi_n^{h,s}(z)$$

$$f_{h,s}(z) = A_{0,s}^h f(z), \quad \Psi_n^{s,s}(z) = A_{n,s}^s\tilde{F}_n(z) + c_s\tilde{H}_n(z), \quad \Psi_n^{k,s}(z) = A_{n,s}^k\tilde{F}_n(z), \quad k \ne s = 0,1$$

Suppose the specified function $\delta(\xi)$ and the required functions $g(\xi)$ and $u(\xi)$ are defined by the series

$$\delta(\xi)=\sum_{k=0}^{\infty}\delta_k\cos a_k\xi,\quad 0\le\xi\le a,\quad a_k=\frac{k\pi}{a};\quad g(\xi)=\sum_{k=0}^{\infty}\delta_k g_k(\xi),\quad a\le\xi\le 1$$

$$g_k(\xi)=X_*^{(k)}+\sum_{h=0}^{2}X_h^{(k)}(\xi-a)^{(h+1)/2}-\sum_{r=1}^{\infty}\frac{X_{r+2}^{(k)}}{l_r^2}\cos l_r(\xi-a),\quad l_r=\frac{r\pi}{l},\quad l=1-a \tag{2.10}$$

$$u(\xi)=\sum_{k=0}^{\infty}\delta_k u^{(k)}(\xi),\quad u^{(k)}(\xi)=\tilde{X}_0^{(k)}+\sum_{r=1}^{\infty}\frac{\tilde{X}_r^{(k)}}{b_r^2}\cos b_r\xi,\quad b_r=r\pi,\quad 0\le\xi\le 1 \tag{2.11}$$

From condition $\delta(a)=g(a)$, we find

$$X_*^{(k)}=(-1)^k+\sum_{r=1}^{\infty}\frac{X_{r+2}^{(k)}}{l_r^2},\quad k=0,1,\ldots$$

$$g_k(\xi)=(-1)^k+\sum_{h=0}^{2}X_h^{(k)}(\xi-a)^{(h+1)/2}+\sum_{r=1}^{\infty}\frac{X_{r+2}^{(k)}}{l_r^2}(1-\cos l_r(\xi-a)),\quad a\le\xi\le 1 \tag{2.12}$$

Substituting expressions (2.10) - (2.12) into Eq. (2.9) and equating the coefficients of $\delta_k(k=0,1,\ldots)$ to zero, we obtain the system of functional equations

$$\sum_{h=0}^{\infty}X_h^{(k)}[f_h^s(z)+j_h f_{0,s}(z)]+\tilde{X}_0^{(k)}f_{1,s}(z)+\sum_{r=1}^{\infty}\tilde{X}_r^{(k)}\tilde{f}_r^s(z)=f^{k,s}(z)-\varepsilon_k f_{0,s}(z) \tag{2.13}$$

when $s=0$, $a\le z\le 1$ and, when $s=1$, $0\le z\le 1$

where

$$f_h^s(z)=\sum_{n=1}^{\infty}Q_{h,n}\Psi_n^{0,s}(z),\quad \tilde{f}_r^s(z)=\sum_{n=1}^{\infty}I_{rn}\Psi_n^{1,s}(z),\quad f^{k,s}(z)=a_k^2\sum_{n=1}^{\infty}\tilde{I}_{kn}\Psi_n^{0,s}(z)$$

$$Q_{q,n}=J_n^{(q)},\quad Q_{r+2,n}=J_{rn},\quad j_q=\int_a^1(\xi-a)^{(q+1)/2}d\xi,\quad j_{r+2}=\frac{l}{l_r^2},\quad \varepsilon_0=1,\quad \varepsilon_r=(-1)^r l$$

$$J_n^{(q)}=\int_a^1(\xi-a)^{(q+1)/2}F_n''(\xi)d\xi,\quad J_{rn}=\frac{1}{l_r^2}\int_a^1 F_n''(\xi)(1-\cos l_r(\xi-a))d\xi$$

$$I_{rn}=\frac{1}{b_r^2}\int_0^1\cos b_r\xi F_n'''(\xi)d\xi,\quad \int_0^a\cos a_k\xi F_n''(\xi)d\xi=(-1)^k F_n'(a)-a_k^2\tilde{I}_{kn}$$

$$q=0,1,2;\quad r=1,2,\ldots;\quad s=0,1 \tag{2.14}$$

Formulae are available for evaluating the integrals $J_n^{(q)}, J_{rn}, J_{rn}, \tilde{I}_{kn}$ [1,2]

It can easily be shown (see Section 3) that the functional series (2.14) converge uniformly in the interval [0,1] and, consequently, they can be integrated term by term. Multiplying Eq. (2.13) by $\cos l_m(z-a)$ when $s=0$ and, by $\cos b_m z (m=0,1,\ldots)$ when $s=1$ and integrating over the intervals $[a, 1]$ and [0,1] respectively, we obtain two infinite systems of algebraic equations in the unknowns $X_h^{(k)}, \widetilde{X}_h^{(k)}\quad (h=0,1,\ldots)$.

$$AX^{(k)}+B\tilde{X}^{(k)}=b^{(k)},\quad \tilde{A}X^{(k)}+\tilde{B}\tilde{X}^{(k)}=\tilde{b}^{(k)}\quad (k=0,1,\ldots) \tag{2.15}$$

Taking the integrals (see Ref. 2)

$$J_m^a=\int_a^1 f(z)\cos l_m(z-a)dz,\quad J_m^0=\int_0^1 f(z)\cos b_m z\,dz,\quad f_{mn}^a=\int_a^1\tilde{F}_n(z)\cos l_m(z-a)dz$$

$$h_{mn}^a=\int_a^1\tilde{H}_n(z)\cos l_m(z-a)dz,\quad f_{mn}^0=\int_0^1\tilde{F}_n(z)\cos b_m z\,dz,\quad h_{mn}^0=\int_0^1\tilde{H}_n(z)\cos b_m z\,dz$$

into account, we obtain expressions for the elements of the matrices A, B, $\widetilde{A}$, $\widetilde{B}$ and the vectors $b^{(k)}$, $\widetilde{b}^{(k)}$

$$a_{m,h} = j_h A^0_{0,0} J^a_m + \sum_{n=1}^{\infty} Q_{h,n}\left(A^0_{n,0} f^a_{mn} + c_0 h^a_{mn}\right), \quad b_{m,0} = A^1_{0,0} J^a_m,$$

$$b_{m,r} = \sum_{n=1}^{\infty} I_{rn} A^1_{n,0} f^a_{mn} \quad \tilde{a}_{m,h} = j_h A^0_{0,1} J^0_m + \sum_{n=1}^{\infty} Q_{h,n} A^0_{n,1} f^0_{mn},$$

$$\tilde{b}_{m,0} = A^1_{0,1} J^0_m, \quad \tilde{b}_{m,r} = \sum_{n=1}^{\infty} I_{rn}\left(A^1_{n,1} f^0_{mn} + c_1 h^0_{mn}\right)$$

$$b^{(k)}_m = a_k^2 \sum_{n=1}^{\infty} \tilde{I}_{kn}\left(A^0_{n,0} f^a_{mn} + c_0 h^a_{mn}\right) - \varepsilon_k A^0_{0,0} J^a_m, \quad \tilde{b}^{(k)}_m = a_k^2 \sum_{n=1}^{\infty} \tilde{I}_{kn} A^0_{n,1} f^0_{mn} - \varepsilon_k A^0_{0,1} J^0_m$$

$$k,m,h = 0,1,\ldots; \quad r = 1,2,\ldots \tag{2.16}$$

The integral equations (2.9) are the consequence of an ill-posed problem because both systems are ill-posed and have to be regularized by introducing the small positive parameters α and $\widetilde{\alpha}$ [3]. The regularized systems have the form

$$\left(A^{\mathrm{T}} A + \alpha E\right) Y^{(k)} + A^{\mathrm{T}} B \tilde{Y}^{(k)} = A^{\mathrm{T}} b^{(k)}, \quad \tilde{B}^{\mathrm{T}} \tilde{A} Y^{(k)} + (\tilde{B}^{\mathrm{T}} \tilde{B} + \tilde{\alpha} E) \tilde{Y}^{(k)} = \tilde{B}^{\mathrm{T}} \tilde{b}^{(k)} \tag{2.17}$$

From here, we determine the regularized solutions $Y^{(k)}$, $\widetilde{Y}^{(k)}$ and the functions

$$g_k(\xi) = (-1)^k + \sum_{h=0}^{2} Y^{(k)}_h (\xi - a)^{(h+1)/2} + 2\sum_{r=1}^{\infty} Y^{(k)}_{r+2}\left(\frac{1}{l_r}\sin\left(l_r \frac{\xi - a}{2}\right)\right)^2 \tag{2.18}$$

$$u^{(k)}(\xi) = \tilde{Y}^{(k)}_0 + \sum_{r=1}^{\infty} \tilde{Y}^{(k)}_r \frac{\cos b_r \xi}{b_r^2}, \quad k = 0,1,\ldots \tag{2.19}$$

Then, using formulae (2.9), we find the functions $\sigma(R_s, z)$ $(s = 0, 1)$, in terms of which the stresses 461 g $\sigma_r(R_S, z) = \sigma'''(R_S, z)$ are expressed. We have

$$\sigma(R_s, z) = \theta \sum_{k=0}^{\infty} \delta_k \sigma_k(R_s, z), \quad \sigma_k(R_s, z) = \alpha^{(k)}_s f(z) + \omega^{(k)}(R_s, z)$$

$$\alpha^{(k)}_s = A^0_{0,s}\left(\varepsilon_k + \sum_{h=0}^{\infty} Y^{(k)}_h j_h\right) + A^1_{0,s} \tilde{Y}^{(k)}_0, \quad \omega^{(k)}(R_s, z) = \sum_{h=0}^{\infty} Y^{(k)}_h f^s_h(z) + \sum_{r=1}^{\infty} \tilde{Y}^{(k)}_r \tilde{f}^s_r(z) - f^{k,s}(z) \tag{2.20}$$

The details of the calculation of the quantity $A^s_{n,s}$ and the hyperbolic functions were presented earlier.[2]

In the case of a plate with a cylindrical cavity ($R_0 = R$, $R_1 = \infty$), the boundary condition (2.9) when $s = 0$, the functional equation (2.13) when $s = 0$ and the systems of algebraic equations (2.15) and (2.17) have the form

$$\sigma(R, z) = \theta\left\{\int_a^1 g(\xi) K(\xi, z) d\xi + \int_0^a \delta(\xi) K(\xi, z) d\xi\right\} = 0, \quad a \le z \le 1 \tag{2.21}$$

$$\sum_{h=0}^{\infty} X^{(k)}_h \left[f_h(z) + j_h \tilde{f}(z)\right] = f^{(k)}(z) - \varepsilon_k \tilde{f}(z), \quad a \le z \le 1 \tag{2.22}$$

$$AX^{(k)} = b^{(k)}, \quad (A^T A + \alpha E) Y^{(k)} = A^T b^{(k)}, \quad k = 0,1,\ldots \tag{2.23}$$

where

$$K(\xi,z)=\tilde{f}(z)+\sum_{n=1}^{\infty}F''_{ni}(\xi)\tilde{\Psi}_n(z),\quad \tilde{\Psi}_n(z)=B_n\tilde{F}_{ni}(z)+c\tilde{H}_{ni}(z),\quad \tilde{f}(z)=\tilde{c}f(z)$$

$$\tilde{F}_{ni}(z)=4t_n^{-1}\left[F'_{ni}(z)+\beta_{ni}(\sin t_n-\sin t_n z)\right],\quad \tilde{H}_{ni}(z)=t_n^{-2}[z-1+\beta_{ni}(\sin t_n-\sin t_n z)]$$

$$c=\frac{4\nu}{R},\quad \tilde{c}=\frac{\nu-1}{R},\quad B_n=A_n+\frac{1}{t_nR},\quad f_h(z)=\sum_{n=1}^{\infty}Q_{h,ni}\tilde{\Psi}_n(z),\quad f^{(k)}(z)=a_k^2\sum_{n=1}^{\infty}\tilde{I}_{k,ni}\tilde{\Psi}_n(z)$$

$$a_{m,h}=J_m^a\tilde{c}j_h+\sum_{n=1}^{\infty}Q_{h,ni}\tilde{J}_{m,ni},\quad b_m^{(k)}=a_k^2\sum_{n=1}^{\infty}\tilde{I}_{k,ni}\tilde{J}_{m,ni}-\varepsilon_kJ_m^a\tilde{c},\quad \tilde{J}_{m,ni}=B_nf_{m,ni}^a+\mathrm{c}h_{m,ni}^a$$

$$f_{m,ni}^a=4\varepsilon'_m\frac{\tan t_n}{t_n^2}+\frac{4t_nF_{ni}(a)}{l_m^2-t_n^2}+4\frac{t_n(-1)^m+\sin t_n\cos t_na}{(l_m^2-t_n^2)^2},\quad \varepsilon'_0=l,\quad \varepsilon'_r=0,\quad r=1,2,\ldots$$

$$h_{m,ni}^a=\varepsilon'_m\frac{\tan t_n}{t_n^3}-\frac{\varepsilon''_m}{t_n^2}+\frac{\cos t_na/\cos t_n-(-1)^m}{t_n^2(l_m^2-t_n^2)},\quad \varepsilon''_0=\frac{l^2}{2},\quad \varepsilon''_r=\frac{1-(-1)^r}{l_r^2},\quad m,h=0,1,\ldots$$

Note that, if $\operatorname{Im}\gamma_n>0$ and $\operatorname{Im}t_n<0$, the relations

$$\gamma_n=it_n,\quad F''_{ni}(z)=F''_{ni}(z),\quad Q_{h,n}=Q_{h,ni},\quad I_{rn}=I_{r,ni},\quad \tilde{I}_{kn}=\tilde{I}_{k,ni},\quad f_{m,ni}^a=if_{mn}^a\ \text{and so on.}$$

hold.

3. Summation of the series

Since the elements of the matrices A, B, ... of Eqs. (2.15) as well as the functions $f_h^s(z),\widetilde{f_r^s}(z),\ldots$, describing the contact stresses are determined by poorly converging numerical and functional series, an effective method was developed for calculating the remainders of the above mentioned series, based on the use of the asymptotic summation formulae

$$J(s,\theta)=\sum_{n=p+1}^{\infty}z^n(\theta)\lambda_n^s,\quad s>0,\quad 0<|\theta|<2;\quad J(s,0)=\sum_{n=p+1}^{\infty}\lambda_n^s,\quad s>1\tag{3.1}$$

Here,

$$\lambda_n=(-4\gamma_n)^{-1},\quad z(\theta)=\exp(i\pi\theta),\quad p\geq 4000.$$

Expressions for $J(s,\theta)$ and $J(s,0)$ were presented earlier.[1,2]

As an example, we will now consider a technique for calculating the values of the function $K(\xi)$

$$K(\xi)=\sum_{n=1}^{p}G_n(\xi)+R_p(\xi),\quad R_p(\xi)=\sum_{n=p+1}^{\infty}G_n(\xi),\quad G_n(\xi)=F''_n(\xi)\operatorname{th}\gamma_n/\gamma_n$$

Here $R_p(\xi)$ is the remainder of the functional series which begins from the $(p+1)$-th term; $|\xi|\leq 1$, $p=4000$.

The sum of the first p terms of the series is calculated directly and the remainder $R_p(\xi)$ is found using the first formula of (3.1) if $|\xi|<1$ and the second formula when $\xi=1$. To do this, the expressions $G_n(\xi)$ and $G_n(1)$ are expanded in series in powers of a small parameter λ_n. It is easy to obtain these expansions using the representation

$$\exp(2\gamma_na)=z^n(2a)\lambda_n^{-a}[1+a\lambda_n^2+a(a-3)\lambda_n^4/2+\ldots]$$

Taking account of this formula, we find

$$G_n(\xi)=q_n(\xi)+q_n(-\xi)|\xi|<1,\quad G_n(1)\equiv\operatorname{th}\gamma_n/\gamma_n=-4\lambda_n+8\lambda_n^2-8\lambda_n^3+\ldots$$
$$q_n(\xi)=z^n(1-\xi)\lambda_n^{(1+\xi)/2}[-(1+\xi)/2+\lambda_n(3\xi-3)/2+\lambda_n^2(23-8\xi+\xi^2)/4+\ldots]\tag{3.2}$$

Then, applying formula (3.1) to each term of expansions (3.2), we finally obtain

$$R_P(\xi) = 2\mathrm{Re}\left\{\sum_{n=p+1}^{\infty}\left[q_n(\xi)+q_n(-\xi)\right]\right\} = \mathrm{Re}\left\{e_p(\xi)+e_p(-\xi)\right\}$$

$$e_p(\xi) = -(1+\xi)J\left(\frac{1+\xi}{2},1-\xi\right)+(3\xi-3)J\left(\frac{3+\xi}{2},1-\xi\right)+\frac{1}{2}(23-8\xi+\xi^2)J\left(\frac{5+\xi}{2},1-\xi\right)$$

$$R_P(1) = 2\mathrm{Re}\left\{\sum_{n=p+1}^{\infty}(-4\lambda_n+8\lambda_n^2-8\lambda_n^3)\right\} = \mathrm{Re}\left\{-8J(1,0)+16J(2,0)-16J(3,0)\right\} \quad (3.3)$$

Note that the quantity $\mathrm{Re}\{J(s, 0)\}$ also exists when $s = 1$.

The results of the calculations using formulae (3.1) and (3.3) are given below.

ξ_k	0	0.2	0.5	0.9	1.0
$R_p(\xi_k)$	$3.15 \cdot 10^{-3}$	$4.02 \cdot 10^{-3}$	$9.32 \cdot 10^{-3}$	$1.46 \cdot 10^{-2}$	$2.74 \cdot 10^{-4}$
$K(\xi_k)$	$-1.39 \cdot 10^{-15}$	$-5.11 \cdot 10^{-13}$	$-3.32 \cdot 10^{-13}$	$-1.58 \cdot 10^{-11}$	$0.5 + 10^{-11}$

From this, taking account of the error in the calculation, we conclude that $K(\xi) = 0|\xi| < 1$, $K(1) = 1/2$. The characteristics of the function $K(\xi)$ which have been noted enable us to simplify the expressions for the functions $\widetilde{F}_n(z)$, $\widetilde{H}_n(z)$ of condition (2.5).

The values of the numerical and functional series in formulae (2.16) and (2.14) are calculated using the same scheme. It is best to use the expansions for the integrals $J_n^{(h)}$, the quantities $A_{n,s}^s$ and the functions $F_n''(\xi)$, $\widetilde{F}_n(z)$, $\widetilde{H}_n(z)$ which have already been found [1,2] in order to obtain expansions in powers of λ_n of the n-th terms of these series. It is then necessary to apply the first formula of (3.1) to the terms of the expansion $A_k z_n(\theta_k)\lambda_n^{s_k}(0 < \theta_k < 2, s_k > 0)$ and the second formula to the terms $B_k\lambda_n^{s_k}(s_k > 1)$.

Among the drawbacks of the method for summing the series described here, we must include the unwieldiness of the expansions used in them and the low accuracy of formula (3.1) when the values of θ are close to 0 and 2 (compare $K(0.9) = -1.58 \cdot 10^{-11}$ and $K(0.99) = -1.55 \cdot 10^{-9}$).

The necessary control on the accuracy in calculating the remainder $R_p = a_{p+1} + a_{p+2} + \ldots$ is achieved using the quantity $\varepsilon_p = |r - R|$ (for an ideal calculation $\varepsilon_p = 0$), where

$$r = R_p - R_{\tilde{p}}, \quad R = a_{p+1} + a_{p+2} + \ldots + a_{\tilde{p}}, \quad \tilde{p} = p + 2000$$

Thus, in checking the remainders $R_p^{(h)}(z)$, $R_p^h(m)$ occurring in the formulae

$$f_h^0(z) = \sum_{n=1}^{p} J_n^{(h)}\Psi_n^{0,0}(z) + R_p^{(h)}(z), \quad a_{m,h} = j_h A_{0,0}^0 J_m^a + \sum_{n=1}^{p} J_n^{(h)}\left(A_{n,0}^0 f_{mn}^a + c_0 h_{mn}^a\right) + R_p^h(m)$$

when $h = m = 0$, $z = 0.5$, the following values are obtained:

$$R_p^{(0)}(0.5) = -2.348 \cdot 10^{-12}, \quad f_0^0(0.5) = 0.0805, \quad \varepsilon_p = 6.5 \cdot 10^{-20}, \quad r = -3.639 \cdot 10^{-12}$$

$$R_p^0(0) = -1.214 \cdot 10^{-12}, \quad a_{0,0} = 0.1108, \quad \varepsilon_p = 2.4 \cdot 10^{-21}, \quad r = 8.384 \cdot 10^{-13}$$

$$(a = R_0 = 0.25, \quad R_1 = 0.5, \quad \nu = 0.3, \quad p = 4000)$$

The kernels $Kh, s(\xi, z)(h, s = 0, 1)$, $K(\xi, z)$ of the integral equations (2.9) and (2.21) are determined by functional series. Summing and then investigating the remainders of these series, it can be successfully established that all the kernels are continuous and bounded in the domain $\bar{D}\{\xi, z \in [0, 1]\}$, and, at the same time, that the kernels $K_{h,h}(\xi, z)$ and $K(\xi, z)$ have a singularity of the $(\xi - z)\ln|\xi - z|$ type in the $D^*\{|\xi - z| \to 0\}$ region. The logarithmic singularity is found taking account of the asymptotic form when $|\xi - z| \to 0$ of the special function $\Phi(z, s, \upsilon)$ [7] in terms of which the remainders investigated are expressed.

4. Determination of the contact pressure

We will now present examples of the calculation of a hollow cylinder with an insert ($\delta(z) \equiv \delta_0$, $k = 0$; $a = R_0 = R = 1/4$) for the following versions: 1) $R_1 = 1/2$, 2) $R_1 = 3.4$, 3) $R_1 = 1$, 4) $R_1 = \infty$ (a plate with a cavity). The infinite systems (2.17) in the unknowns $Y_s^{(0)}$, $\widetilde{Y}_s^{(0)}(s = 0, 1, \ldots)$ (we will henceforth omit the zero superscript on the quantities $Y_s^{(0)}$, $u^{(0)}(z), \ldots$) were shortened and solved for several values of α and $\widetilde{\alpha}$. Its own pair $(\alpha, \widetilde{\alpha})$ of smallest values of the regularization parameters for which no noticeable amplitudes of the oscillations of the regularized solutions Y_s, $\widetilde{Y}_s(s = 0, \ldots, 80)$ were yet observed and the discrepancy was fairly small

$$|\,\sigma_0(R_1, z)\,| < 6 \cdot 10^{-9}, 0 \le z < 1; \quad |\,\sigma_0(R_0, z)\,| < 4 \cdot 10^{-9}, a < z < 1$$

was chosen for each version (the values of the pairs were: $(8 \cdot 10^{-19}, 2 \cdot 10^{-19})$, $(4 \cdot 10^{-19}, 2 \cdot 10^{-19})$ for versions 1, 2, 3 and 4 respectively).

The search for the optimal pair $(\alpha, \widetilde{\alpha})$ is helped considerably by the fact that the appearance of noticeable amplitudes of the the oscillations of the solutions $\widetilde{Y}_s$ and $\widetilde{Y}_s$ is determined by one parameter: α or $\widetilde{\alpha}$ respectively.

Table 1

s	$Y_s 10^5$				$Y_s 10^5$		
	Versions						
	1	2	3	4	1	2	3
0	−215767	−240330	−265767	−284565	23130	17181	13405
1	54035	104363	203983	219228	366870	237813	146127
2	42460	−9622	−68 847	−59 923	459 289	74164	−22 262
3	120 979	207 987	134 092	167 774	50446	94 628	67 455
75	17	−1016	−197	−302	14805	49890	40 507
76	−63	755	5	40	−14672	−48 873	−39525
77	28	−925	−188	−290	13631	46048	37418
78	−55	693	7	19	−13764	−45698	−36915
79	14	−865	−182	−282	12620	42779	34802
80	−41	645	−24	−8	−13689	−45102	−36337

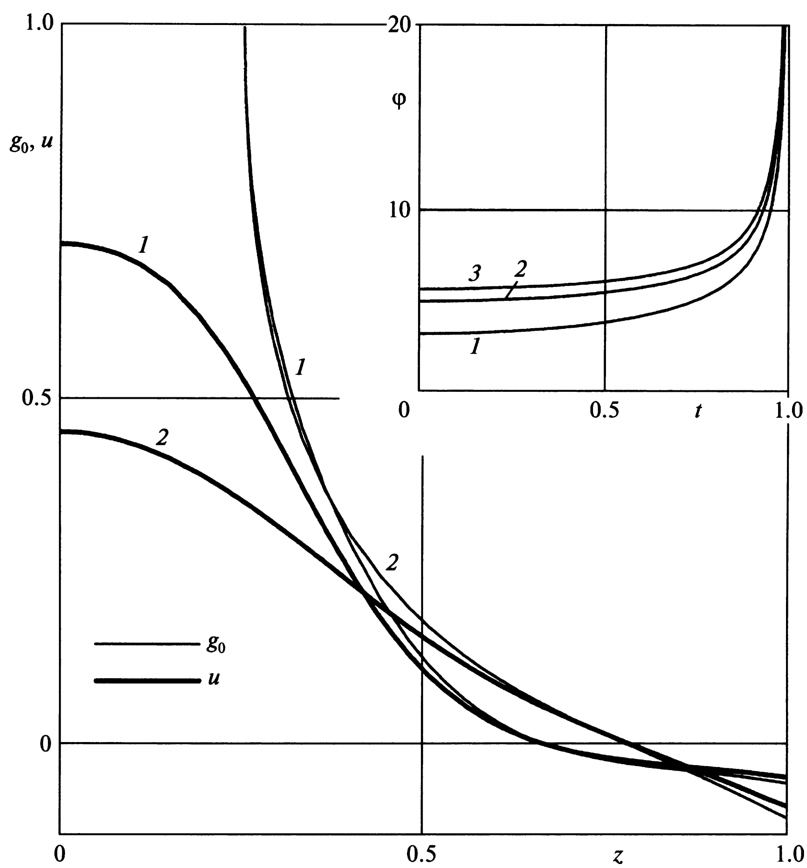

Fig. 2.

The values of the constants $Y_s \times 10^5$ and $\widetilde{Y}s \times 10^5 (s = 0, \ldots, 3; s = 75, \ldots 80)$ are given in Table 1.

Graphs of the functions $g_0(z)$ and $u^{(0)}(z) \equiv u(z)$, obtained using formulae (2.18) and (2.19), are shown in Fig. 2. The number on a curve corresponds to the version number.

In order to find the contact pressure $q(z) = -\sigma_r(R_0, z)(|z| \leq a)$, we turn to relations (2.20) when $k = 0$

$$\sigma(R_0, z) = \theta\delta_0\sigma_0(R_0, z), \quad \sigma_0(R_0, z) = \alpha_0 f(z) + \omega(R_0, z), \quad \sigma_r(R_0, z) = \theta\delta_0\sigma_0'''(R_0, z)$$

$$\alpha_0 = A_{0,0}^0\left(1 + \sum_{h=0}^{80} Y_h j_h\right) + A_{0,0}^1 \tilde{Y}_0, \quad \omega(R_0, z) = \sum_{h=0}^{80} Y_h f_h^0(z) + \sum_{r=1}^{80} \tilde{Y}_r \tilde{f}_r^0(z)$$

Table 2

k	$\varphi(t_k)$			
	Versions			
	1	2	3	4
0	3.186	4.965	5.620	6.213
1	3.242	5.005	5.654	6.246
2	3.424	5.134	5.766	6.356
	3.780	5.394	6.002	6.590
4	4.441	5.913	6.497	7.093
5	5.939	7.255	7.848	8.501

Table 3

Quantity	Versions				
	1	2	3	4	5
χ_1	2.526	3.335	3.660	3.994	3.860
χ_2	0.797	1.241	1.405	1.553	1.480
χ_3	0.766	0.852	0.913	0.979	0.937
$u(0)$	0.7013	0.4379	0.2813	–	–
$u(1)$	−0.0447	−0.0852	−0.0461	–	–
$g_0(1)$	−0.0531	−0.1011	−0.0597	−0.0105	–

In the case of a plate with a cavity, we have

$$\sigma_r(R,z) = \theta\delta_0\sigma_0'''(R,z), \quad \sigma_0(R,z) = \alpha\tilde{f}(z) + \omega(R,z)$$

$$\alpha = 1 + \sum_{h=0}^{80} Y_h j_h, \quad \omega(R,z) = \sum_{h=0}^{80} Y_h f_h(z)$$

It follows from this that the dimensionless contact pressure distribution function $\tilde{\varphi}(z)$ and the integral characteristic N_0 are defiined by the expressions

$$\tilde{\varphi}(z) = q(z)(\theta\delta_0)^{-1} = -\sigma_0'''(R_0,z) = -2\alpha_0 - \omega'''(R_0,z)$$

$$aN_0 = -2\int_0^a \sigma_0'''(R_0,z)dz = -2\sigma_0''(R_0,a) - 4\alpha_0 + 2\sum_{h=0}^{80} Y_h\left(f_h^0(0)\right)'' + 2\sum_{r=1}^{80}\tilde{Y}_r\left(\tilde{f}_r^0(0)\right)'' \tag{4.1}$$

and, in the case of a plate with a cavity, by

$$\tilde{\varphi}(z) = 2\alpha(1-\nu)R^{-1} - \omega'''(R,z)$$

$$aN_0 = -2\sigma_0''(R,a) + 4\alpha(1-\nu)R^{-1} + 2\sum_{h=0}^{80} Y_h f_h''(0) \tag{4.2}$$

Taking account of the equality

$$\sigma_0''(R_0,a) = \sigma_0''(R,a) = \left(f_h^0(0)\right)'' = \left(\tilde{f}_r^0(0)\right)'' = f_h''(0) = 0$$

we find the formulae for the integral characteristics of the hollow cylinder and the plate

$$N_0 = -4a^{-1}\alpha_0, \quad N_0 = 4\alpha(1-\nu)(aR)^{-1}$$

The third derivative $\tilde{\omega}'''(R_0, z)$ at the central mesh point $z = z_0$ is found numerically [3,6]

$$\omega'''(R_0,z_0) = h^{-3}\left(\frac{1}{8}\omega_{-3} - \omega_{-2} + \frac{13}{8}\omega_{-1} - \frac{13}{8}\omega_1 + \omega_2 - \frac{1}{8}\omega_3\right) + O(h^4)$$

Here, $\omega_k = \omega(R_0, z_k)$, $z_k = z_0 + kh$, $k = -3, \ldots, 3$; $0005 \le h \le 0.001$.
The values of the function $\varphi(t) \equiv \tilde{\varphi}(at) \quad (t = z/a)$ when $t = t_k = k/6$ are presented in Table 2, and the values of the quantities $(\nu = 0.3)$

$$\chi_1 = aN_0, \quad \chi_2 = a\varphi(0), \quad \chi_3 = a\lim\varphi(t)\sqrt{1-t^2} \quad (t \to 1)$$

and the functions $u(z)$, $g_0(z)$ when $z = 0$ and $z = 1$ are presented in Table 3. Comparing the values of $\chi_r (r = 1, 2, 3)$ for hollow cylinders of finite dimensions and an unbounded plate with a cylindrical cavity with the corresponding values of χ_r for a space with a cylindrical cavity (Version 5, see Ref. 4, p. 97), we see that they differ by less than 4.9% (Version 4), 5.2% (Version 3), 16% (Version 2) and 46% (Version 1).

Graphs of the function $\varphi(t)$, obtained using formula (4.1), are shown in the right upper part of Fig. 2. In order to explain these graphs, we separate out the root singularity of the function $\sigma_r(R_0, z)$, for example (Version 3), in the left half-neighbourhood of the point $z = a$:

$$\sigma_r(R_0,z) = -\theta\delta_0(a-z)^{-1/2}L_4(z), \quad 0 \le a-z \le 28\tilde{h}; \quad \sigma_r(R_0,z) = 0, \quad z > a$$

Here, $L_4(z) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ is a generalized interpolation polynomial for the function $y(z) = x\widetilde{\varphi}(z)(x \equiv \sqrt{a-z})$ which is given at the interpolation points

$$x_k = (k+1)\sqrt{\tilde{h}}, \quad z_k = a-(k+1)^2\tilde{h}, \quad k = 0,\ldots,4; \quad \tilde{h} = 0.0025.$$

Calculating the values of $y_k = y(z_k)$ when $h = 0.0005(k = 0)$ and $h = 0.0006(k = 1, \ldots 4)$

$$y_0 = 1.29713, \quad y_1 = 1.35942, \quad y_2 = 1.46030, \quad y_3 = 1.59007, \quad y_4 = 1.74323$$

and then the coefficients $a_k(k = 0, \ldots, 4)$, we find

$$\tilde{\varphi}(z) = \frac{1}{\sqrt{a-z}}\sum_{k=0}^{4} a_k(a-z)^{k/2} + \varepsilon(z), \quad \tilde{h} \le a-z \le 28\tilde{h}, \quad |\varepsilon(z)| \le 4\cdot 10^{-4}$$

$$a_0 = 1.28734, \quad a_1 = -0.44296, \quad a_2 = 14.053, \quad a_3 = -26.957, \quad a_4 = 28.040$$

Similar calculations carried out for a plate with a cavity give

$$\tilde{\varphi}(z) = \frac{1}{\sqrt{a-z}}\sum_{k=0}^{4} b_k(a-z)^{k/2} + \tilde{\varepsilon}(z), \quad \tilde{h} \le a-z \le 28\tilde{h}, \quad |\tilde{\varepsilon}(z)| \le 5\cdot 10^{-4}$$

$$b_0 = 1.37828, \quad b_1 = -0.45572, \quad b_2 = 15.3561, \quad b_3 = -28.2850, \quad b_4 = 29.2300$$

Note the good agreement of the quantities $a_0/\sqrt{2} = 0.910$, $b_0/\sqrt{2} = 0.975$ $\chi_3 = 0.913$, $\chi_3 = 0.979$ (see Table 3, versions 3 and 4).

References

1. Aleksandrov VM, Bazarenko NA. The contact problem for a rectangle with stress-free side faces. *Prikl Mat Mekh* 2007;**71**(2):340–51.
2. Bazarenko NA. The contact problem for hollow and solid cylinders with stress-free faces. *Prikl Mat Mekh* 2008;**72**(2):328–41.
3. Kalitkin NN. *Numerical Methods Moscow:Nauka* 1978.
4. Aleksandrov VM, Pozharskii DA. *Non-Classical Spatial Problems in the Mechanics of Contact Interactions of Elastic Bodies*. Moscow:Faktorial;1998.
5. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*/Eds M. Abramowitz and I. Stegun. Washington: Gov. Print off. 1964.
6. Danilina NI, Dubrovskaya NS, Kvasha OP et al. *Numerical Methods. A Textbook for Special Educational Institutions* Moscow:Vyssh. Shkola;1976.
7. Bateman H, Erdelyi A. *Higher Transcendental Function N.Y.*. McGraw Hill; 1995.

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